Local Quantum Uncertainty and Bounds on Quantumness for $\mathcal{O} \otimes \mathcal{O}$ invariant class of states

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We derive closed form of local quantum uncertainty and bounds for post-entanglement correlation measures- geometric discord and measurement-induced nonlocality for highly symmetric orthogonal invariant sates. This class of states includes both the Werner and Isotropic class. We provide exact formula for local quantum uncertainty for two-qutrit systems. We also provide a comparative study of the upper and lower bounds of geometric discord with entanglement, as measured by negativity for a subclass of states not included in both the Werner and Isotropic class of states.

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I. INTRODUCTION

Quantum mechanics shows several counterintuitive results when we are dealing with composite systems [1–3]. There exist peculiar type of correlations between different parts of a composite system commonly known as nonclassical correlations. Entanglement is one of the most powerful nonclassical correlation that establishes its importance in different information processing tasks. However, several post entanglement correlation measures have generated a lot of interests in recent years. Discord, quantum deficit, measurement-induced nonlocality(in short, MIN) [4–7] are a few of them. Even there are different non equivalent versions of discord [8]. Recently Girolami *et. al.*[10] introduced the concept of local quantum uncertainty which quantifies the uncertainty in a quantum state due to measurement of a local observable. Nevertheless, such quantifier has strong reasons to be considered as a faithful measure of quantumness in quantum states. But due to inherent optimization, finding closed formula is a difficult problem for most of the correlations measures. The value of quantum discord is not even known for general bipartite qubit system. In higher dimensional bipartite systems, the results are known for only some special classes of states [11, 12]. Geometric discord has explicit formula for qubit-qudit system and its lower bound is calculated in [13, 14] for higher dimensions. MIN has a closed formula for qubit-qudit systems and it has tight upper bound in higher dimensions [6]. It is possible to derive closed formula for MIN, geometric discord and also for quantum discord for Werner and Isotropic classes of states due to their highly inherent symmetry in the structures. However, local quantum uncertainty (LQU) has closed form only for any qubit-qudit system.

Here, we will consider orthogonal invariant class of states which is a larger class of symmetric states and it contains both Werner and Isotropic classes. We will derive closed form of LQU for this class of states in two qudit system. We will also evaluate bounds of geometric discord and MIN for this symmetric class of states and compare it with an entanglement measure, viz., negativity, for a special subclass of states.

Our work is organized as follows: In the next section we will discuss the concept of local quantum uncertainty and its relevant properties. In section III we will discuss the general properties of generators of SU(n). This will help us to evaluate the correlation measures in higher dimension. Then we will discuss the notions of discord and MIN in section IV. Section V contains a brief discussion on $\mathcal{O} \otimes \mathcal{O}$ invariant class of states and calculation of LQU for that class. In section VI, we will derive bounds of discord and MIN for the class of state, introduced in penultimate section. Section VII contains the comparative study on geometric discord and negativity for a subclass of $\mathcal{O} \otimes \mathcal{O}$ invariant class of states and we conclude in section VIII.

II. LOCAL QUANTUM UNCERTAINTY

Classically, it is possible to measure any two observable with arbitrary accuracy. However, such measurement is not always possible in quantum systems. Uncertainty relation gives the statistical nature of errors in these kind of measurement. Measurement of single observable can also help to detect uncertainty of a quantum observable. For a quantum state ρ, an observable is called *quantum certain* if the error in measurement of the observable is due to only the ignorance about the classical mixing in ρ. A good quantifier of this uncertainty of an observable is the skew information, defined by Wigner and Yanase [9] as

$$I(\rho, K) := -\frac{1}{2}\mathrm{tr}\{[\sqrt{\rho}, K^A]^2\} \tag{1}$$

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Wigner and Yanase introduced this quantity as a measure of information content of the ensemble ρ_{AB} skew to a fixed conserved quantity K^A. Since it quantifies non-commutativity between a quantum state and an observable so it serves as a measure of uncertainty of the observable K^A in the state ρ_{AB}. This type of measure helps to quantify the quantum part of error in measuring an observable. $I = 0$ indicates quantum certain nature of the observable K^A. For a bipartite quantum state ρ_{AB}, Girolami *et.al.* [10] introduced the concept of local quantum uncertainty(LQU) and it is defined as

$$\mathcal{U}_A := \min_{K^A} I(\rho_{AB}, K^A) \tag{2}$$

The minimization is performed over all local *maximally informative observable* (or non-degenerate) $K^A = K_A \otimes \mathbb{I}$. This quantity quantifies the minimum amount of uncertainty in a quantum state. Non-zero value of this quantity indicates the non existence of any quantum certain observable for the state ρ_{AB}. This quantity possess many interesting properties, such as: a) it vanishes for all zero discord state w.r.t. measurement on party A; b) it is invariant under local unitary; c) it reduces to entanglement monotone for pure state. In fact, for pure bipartite states it reduces to linear entropy of reduced subsystems. So, LQU can be taken as a measure of bipartite quantumness. d) For any two observable K_1^A, K_2^A on party A, $\mathrm{Var}_\rho(K_1^A)\mathrm{Var}_\rho(K_2^A) \geq \mathcal{U}_A^2(\rho)$ holds. So, $\mathcal{U}_A^2(\rho) \geq 0$ is an important relation since it connects uncertainty with quantum correlation in observable-independent way. LQU is believed to be the reason behind quantum advantage in DQC1 model and it also works as a lower bound of quantum Fisher Information in parameter estimation. LQU is inherently an asymmetric quantity and explicit closed form is available only for qubit-qudit system. For a quantum state ρ of $2\otimes n$ system, LQU reduces to $1-\lambda_{max}(\mathcal{W})$ where λ_{max} is the maximum eigenvalue of the matrix $\mathcal{W} = (w_{ij})_{3\times3}$, $w_{ij} = \mathrm{tr}\{\sqrt{\rho}(\lambda_i\otimes\mathbb{I})\sqrt{\rho}(\lambda_j\otimes\mathbb{I})\}$ and λ_i's are standard Pauli matrices in this case.

III. GENERATORS OF SU(n) AND THEIR ALGEBRA

Any state of a $n \otimes n$ quantum system can be written in general, as of the form:

$$\rho = \frac{1}{n^2}[\mathbb{I}_n \otimes \mathbb{I}_n + \mathbf{x^t}\lambda \otimes \mathbb{I}_n + \mathbb{I}_n \otimes \mathbf{y^t}\lambda + \sum t_{ij}\lambda_i \otimes \lambda_j] \tag{3}$$

where $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{n^2-1})^t$ and λ_i's are the generators of SU(n). For $n = 2$, Pauli matrices can be used as the generators of SU(2). While for $n = 3$, generallly, Gell-Mann matrices are taken as the generators of SU(3). In this way we can construct traceless, orthogonal generators (*generalized Gell-Mann matrices*) for SU(n), containing $n^2 - 1$ elements as:

$$\lambda_\alpha = \begin{cases} \sqrt{\frac{2}{\alpha(\alpha+1)}}\left(\sum_{k=1}^{\alpha}|k\rangle\langle k| - \alpha|\alpha+1\rangle\langle\alpha+1|\right), & \alpha = 1, ..., n-1 \\ |k\rangle\langle m| + |m\rangle\langle k|, & 1 \leq k < m \leq n, \alpha = n, ..., \frac{n^2+n}{2} - 1 \\ \mathrm{i}(|k\rangle\langle m| - |m\rangle\langle k|), & 1 \leq k < m \leq n, \alpha = \frac{n(n+1)}{2}, ..., n^2 - 1 \end{cases} \tag{4}$$

Among the $(n^2 - 1)$ matrices, the first $(n - 1)$ are mutually commutative, next $(n^2 - 1)/2$ are symmetric and rest $(n^2 - 1)/2$ are antisymmetric. The generators λ_α satisfy the orthogonality relation $\mathrm{tr}(\lambda_\alpha\lambda_\beta) = 2\delta_{\alpha\beta}$. The generators satisfy the following commutation and anti-commutation relations,

$$\begin{aligned} [\lambda_i,\ \lambda_j] =& 2\mathrm{i}\sum_k f_{ijk}\,\lambda_k \\ \{\lambda_i,\ \lambda_j\} =& 2\sum_k d_{ijk}\,\lambda_k + \frac{4}{n}\delta_{ij}\,\mathbb{I}_n \end{aligned} \tag{5}$$

$\mathbb{I}_n$ is identity matrix of dimension n. f_{ijk} are real anti-symmetric tensors and d_{ijk} are real symmetric tensors. They are the structure constants of SU(n). They are determined by the relations,

$$\begin{aligned} f_{ijk} &:= \frac{1}{4\mathrm{i}}\mathrm{tr}([\lambda_i, \lambda_j]\lambda_k) \\ d_{ijk} &:= \frac{1}{4}\mathrm{tr}(\{\lambda_i, \lambda_j\}\lambda_k) \end{aligned} \tag{6}$$

From the relations (5) it follows,

$$\lambda_i\,\lambda_j = \mathrm{i}\sum_k f_{ijk}\,\lambda_k + \sum_k d_{ijk}\,\lambda_k + \frac{2}{n}\delta_{ij}\,\mathbb{I}_n \tag{7}$$

IV. GEOMETRIC DISCORD AND MEASUREMENT-INDUCED NONLOCALITY IN BRIEF

Suppose ρ be any bipartite state shared between two parties, say, Alice(A) and Bob(B). Geometric discord for

the quantum state ρ is defined [5] as the distance from its nearest classical-quantum state, i.e.,

$$D(\rho) = \min_{\chi \in \Omega_0} \parallel \rho - \chi \parallel^2 \tag{8}$$

where Ω_0 is the set of all classical-quantum (zero discord) states and $\parallel . \parallel$ is the usual Hilbert-Schmidt norm (i.e., $\parallel X \parallel := [\mathrm{tr}(X^\dagger X)]^{\frac{1}{2}}$). Since Ω_0 is not convex it is hard to perform the optimization problem in general. Luo *et.al.*[15] introduced an equivalent definition of geometric discord in terms of von Neumann measurements on ρ_A

$$D(\rho) = \min_{\Pi^A} \parallel \rho - \Pi^A(\rho) \parallel^2 \tag{9}$$

Another post entanglement quantum correlation measure is measurement-induced nonlocality and is defined by[6],

$$N(\rho) := \max_{\Pi^A} \parallel \rho - \Pi^A(\rho) \parallel^2 \tag{10}$$

where maximum is taken over all von Neumann measurements Π^A which do not disturb ρ_A, the local density matrix of A, i.e.,

$$\Sigma_k \Pi_k^A \rho_A \Pi_k^A = \rho_A \tag{11}$$

and $\parallel . \parallel$ is taken as the Hilbert Schmidt norm. It quantifies the global disturbance caused by the locally invariant measurement. Both the definitions are not symmetric, i.e., they depend on the the party on which measurement is performed.

Both geometric discord and MIN have closed form in qubit-qudit system. For higher dimensions, the above optimization problems are tackled in [13]. It turns out that the optimization problem (9) should satisfy four constraints (10a-10d)(refer [13]) while the optimization problem (10) should satisfy another extra constraint (11). Based on the optimization, bounds for geometric discord and MIN have been found for a general bipartite state. For a bipartite state ρ in $n \otimes n$ scenario the bounds turn out as,

$$D(\rho) \geq \frac{1}{n^2}[\frac{2}{n}||\mathbf{x}||^2 + \frac{4}{n^2}||T||^2 - \sum_{k=1}^{n-1} \alpha_k^{\downarrow}] \tag{12}$$

where $T = (t_{ij})_{n^2-1 \times n^2-1}$ and $\alpha_k^{\downarrow}$'s are the eigenvalues of $G_1 := \frac{2}{n}\mathbf{x}\mathbf{x}^t + \frac{4}{n^2}TT^t$ in non-increasing order, and

$$N(\rho) \leq \frac{4}{n^4} \sum_{k=1}^{n^2-n} \beta_k^{\downarrow} \tag{13}$$

where $\beta_k^{\downarrow}$'s are the eigenvalues of $G_2 := TT^t$ in non-increasing order.

V. $\mathcal{O} \otimes \mathcal{O}$ INVARIANT CLASS OF STATES AND LOCAL QUANTUM UNCERTAINTY

Consider the group $G = \{\mathcal{O} \otimes \mathcal{O} : \mathcal{O}$ is any orthogonal matrix$\}$. The commutant G' of the group G contains the class of $\mathcal{O} \otimes \mathcal{O}$ invariant states. The commutant is spanned by the three operators $\mathbb{I}$, $\mathbb{F}$, $\hat{\mathbb{F}}$. $\mathbb{I}$ is the identity operator, $\mathbb{F}$ is the flip operator which has the operator form

$$\mathbb{F} = \sum_{i,j} |i\, j\rangle\langle j\, i| \tag{14}$$

$\hat{\mathbb{F}}$ is the projection on maximally entangled state and it has the operator form

$$\hat{\mathbb{F}} = \sum_{i,j} |i\, i\rangle\langle j\, j| \tag{15}$$

The operators satisfy the algebra $\mathbb{F}^2 = \mathbb{I}, \mathbb{F}\hat{\mathbb{F}} = \hat{\mathbb{F}}\mathbb{F} = \hat{\mathbb{F}}, \hat{\mathbb{F}}^2 = n\,\hat{\mathbb{F}}$, n is the dimension of each subsystem.

Any operator from the commutant G' can be written as a linear combination of the three operators $\mathbb{I}, \mathbb{F}, \hat{\mathbb{F}}$. To be a legitimate state the operator should satisfy some other conditions. Consider a $n \times n$ state $\rho \in G'$

$$\rho = a\,\mathbb{I}_{n^2} + b\,\mathbb{F} + c\,\hat{\mathbb{F}} \tag{16}$$

with $n(na + b + c) = 1$ (trace condition) and proper positivity constraints.

The parametrization procedure can be done in another way by considering the expectation values of the operators $\mathbb{I}_n$, $\mathbb{F}$, $\hat{\mathbb{F}}$ [16]. Expectation value of $\mathbb{I}_n$ just gives the relation $\mathrm{tr}\,\rho = 1$ which is obvious. We define two parameters f and $\hat{f}$ as

$$\begin{aligned} f &:= \langle \mathbb{F} \rangle_\rho = \mathrm{tr}(\rho\mathbb{F}) \\ \hat{f} &:= \langle \hat{\mathbb{F}} \rangle_\rho = \mathrm{tr}(\rho\hat{\mathbb{F}}) \end{aligned} \tag{17}$$

As in [16], we can define three orthogonal projectors U, V and W as,

$$\begin{aligned} U &= \hat{\mathbb{F}}/n \\ V &= (\mathbb{I}_{n^2} - \mathbb{F})/2 \\ W &= (\mathbb{I}_{n^2} + \mathbb{F})/2 - \hat{\mathbb{F}}/n \end{aligned}$$

In terms of this orthogonal basis, ρ can be expressed as,

$$\rho = \frac{\hat{f}}{n}U + \frac{1-f}{n(n-1)}V + \frac{n+nf-2\hat{f}}{n(n-1)(n+2)}W \tag{18}$$

The old parameters a, b, c are connected to the new ones f, $\hat{f}$ by the relation,

$$\begin{pmatrix} 1 \\ f \\ \hat{f} \end{pmatrix} = n \begin{pmatrix} n & 1 & 1 \\ 1 & n & 1 \\ 1 & 1 & n \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

In terms of the new parameters the positivity conditions on ρ reads,

$$\begin{aligned} &0 \le \hat{f} \\ &f \le 1 \\ &\hat{f} \le n(f+1)/2 \end{aligned} \tag{19}$$

This is an important class of states of bipartite systems. This class can have both PPT(positive partial transpose) and NPT(negative partial transpose) states depending on the extra constraints on the parameters. When $b = c$ (equivalently $f = \hat{f}$) the positivity conditions of ρ implies the corresponding positivity of partial transpositions ρ^{T_A} or ρ^{T_B}. In case of $b \neq c$ we can find NPT states. Now, $\sqrt{\rho}$ can expressed as,

$$\begin{aligned} \sqrt{\rho} =& \sqrt{\frac{\hat{f}}{n}} U + \sqrt{\frac{1-f}{n(n-1)}} V + \sqrt{\frac{n+nf-2\hat{f}}{n(n-1)(n+2)}} W \\ =& a_1 \, \mathbb{I}_{n^2} + b_1 \, \mathbb{F} + c_1 \, \hat{\mathbb{F}} \end{aligned} \tag{20}$$

with

$$\begin{aligned} a_1 =& \frac{1}{2}\left(\sqrt{\frac{1-f}{n(n-1)}} + \sqrt{\frac{n+nf-2\hat{f}}{n(n-1)(n+2)}}\right) \\ b_1 =& \frac{1}{2}\left(\sqrt{\frac{n+nf-2\hat{f}}{n(n-1)(n+2)}} - \sqrt{\frac{1-f}{n(n-1)}}\right) \\ c_1 =& \frac{1}{n}\left(\sqrt{\frac{\hat{f}}{n}} - \sqrt{\frac{n+nf-2\hat{f}}{n(n-1)(n+2)}}\right) \end{aligned} \tag{21}$$

We can choose any A-observable K_A=$\mathbf{s}.\lambda$ with $\mathbf{s} = (s_1, s_2, ..., s_{n^2-1})$, $|\mathbf{s}| = 1$ and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{n^2-1})^t$. From the definition of local quantum uncertainty(LQU), we have,

$$\begin{aligned} \mathcal{U}_A =& \min I(\rho, K^A) \\ =& \min \{\mathrm{tr}(\rho (K^A)^2) - \mathrm{tr}(\sqrt{\rho} K^A \sqrt{\rho} K^A)\} \\ =& \min\{\mathrm{tr}\{\rho(\mathbf{s}.\lambda \otimes \mathbb{I}_n)^2\} - \mathrm{tr}\{\sqrt{\rho}(\mathbf{s}.\lambda \otimes \mathbb{I}_n)\sqrt{\rho}(\mathbf{s}.\lambda \otimes \mathbb{I}_n)\}\} \end{aligned} \tag{22}$$

using the relation (7) the first term in the minimization problem reduces to,

$$\sum_{i,j,k} s_i s_j [(\mathrm{i} f_{ijk} + d_{ijk})\mathrm{tr}(\rho\lambda_k \otimes \mathbb{I}_n) + \frac{2}{n}\delta_{ij}\mathrm{tr}(\rho)] \tag{23}$$

Since, for any $\mathcal{O} \otimes \mathcal{O}$ invariant state $\mathrm{tr}(\rho\lambda_k \otimes \mathbb{I}_n) = 0$ (30), this term reduces to $\frac{2}{n}$. The second term inside the minimization can be expressed as,

$$\sum_{ij} s_i s_j \mathrm{tr}\{\sqrt{\rho}(\lambda_i \otimes \mathbb{I}_n)\sqrt{\rho}(\lambda_j \otimes \mathbb{I}_n)\} = \mathbf{s}.\mathcal{W}.\mathbf{s}^\dagger$$

The matrix $\mathcal{W}$ is defined as $\mathcal{W} = (w_{ij})$, $w_{ij} = \mathrm{tr}\{\sqrt{\rho}(\lambda_i \otimes \mathbb{I}_n)\sqrt{\rho}(\lambda_j \otimes \mathbb{I}_n)\}$. Hence we can simplify the value of $\mathcal{U}_A$ in terms of maximum eigenvalue λ_{max} of $\mathcal{W}$ as

$$\mathcal{U}_A = \frac{2}{n} - \lambda_{max}(\mathcal{W}) \tag{24}$$

It is clear that the above result (24) also holds for the large class of states with $\mathrm{tr}(\rho\lambda_i \otimes \mathbb{I}_n) = 0$, $i = 1, 2, ..., n^2 - 1$. Hence, closed form of LQU is possible for a large class of bipartite states, depending on the previous condition. Here, we will deal with the orthogonal invariant class of states for our purpose. However, for qubit-qudit system(with observable on the qubit system), $d_{ijk} = 0$ and antisymmetry of f_{ijk} implies $\sum_{i,j,k} s_i s_j f_{ijk} \mathrm{tr}(\rho\lambda_k \otimes \mathbb{I}_n) = 0$. This recovers the result of [10].

For $\mathcal{O} \otimes \mathcal{O}$ invariant state, $\mathcal{W}$ is diagonal in form and its eigenvalues are $2(na_1^2 \pm 2b_1c_1 + 2a_1b_1 + 2a_1c_1)$. Hence LQU can be obtained easily from equation (24). Particularly for two-qutrit system, $\mathcal{W}$ has two distinct eigenvalues $2(3a_1^2 \pm 2b_1c_1 + 2a_1b_1 + 2a_1c_1)$. Hence in this case

$$\mathcal{U}_A = \begin{cases} \frac{2}{3} - 2(3a_1^2 + 2b_1c_1 + 2a_1b_1 + 2a_1c_1), & b_1c_1 \ge 0 \\ \frac{2}{3} - 2(3a_1^2 - 2b_1c_1 + 2a_1b_1 + 2a_1c_1), & b_1c_1 < 0 \end{cases} \tag{25}$$

The corresponding regions are plotted in FIG. 1 for two-qutrit system. For Werner ($c = 0$) and Isotropic ($b = 0$) class of states in two-qutrit system, the eigenvalues of $\mathcal{W}$ become all equal. Hence the explicit form of LQU are,

$$\begin{aligned} \mathcal{U}_A(\rho^{wer}) =& \frac{1}{3}(1 - \sqrt{1-12b}\sqrt{1+6b}) - b \\ \mathcal{U}_A(\rho^{iso}) =& \frac{4}{27}(1 - \sqrt{1-3c}\sqrt{1+24c}) + \frac{14}{9}c \end{aligned} \tag{26}$$

We have shown the nature of LQU for these two classes in FIG. 2 and 3.

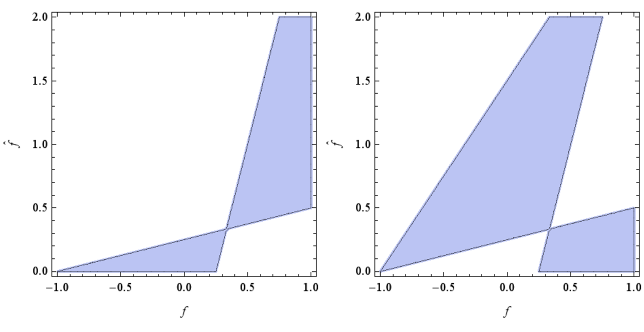

FIG. 1: (Color online) *Region plot in f $\hat{f}$-plane of the eigenvalue of W in two-qutrit system. Both the regions are enclosed by the constraints (19). First figure shows the region (colored) where $b_1c_1 \ge 0$ and Second one shows the region (colored) where $b_1c_1 < 0$. Hence in the first region* $\mathcal{U}_A = \frac{2}{3} - 2(3a_1^2 + 2b_1c_1 + 2a_1b_1 + 2a_1c_1)$ *and in the second* $\mathcal{U}_A = \frac{2}{3} - 2(3a_1^2 - 2b_1c_1 + 2a_1b_1 + 2a_1c_1)$

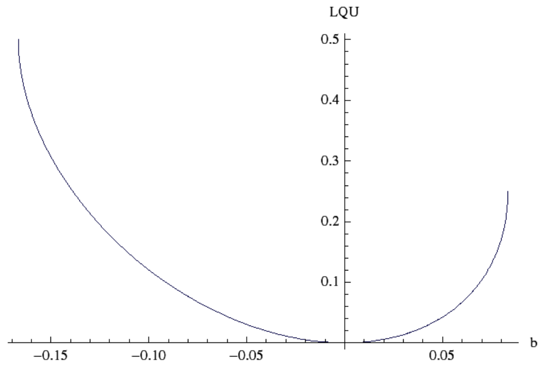

FIG. 2: *LQU for Werner class of states in two-qutrit system for suitable parameter range of b. The class is obtained by putting $c = 0$ in (16). The highest value of $\mathcal{U}_A$ reaches 0.5.*

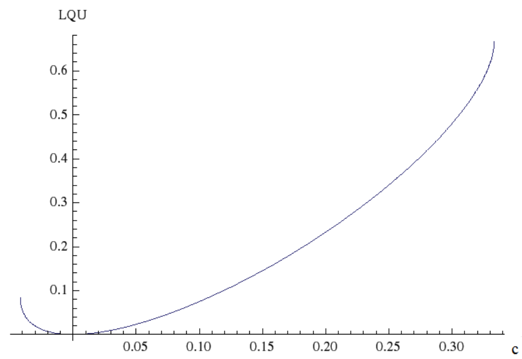

FIG. 3: *LQU for Isotropic class of states in two-qutrit system for suitable range of the parameter c. The class is obtained by putting $b = 0$ in (16). The highest value of $\mathcal{U}_A$ reaches 0.66 in this case.*

VI. BOUNDS FOR DISCORD AND MIN FOR $\mathcal{O} \otimes \mathcal{O}$ INVARIANT STATES

For the state ρ in (3), we have the Bloch vector $\mathbf{x} = (x_1, x_2, ..., x_n)^t$ with $x_k = \frac{n}{2}\mathrm{tr}(\rho\lambda_k \otimes \mathbb{I}_n)$ and correlation matrix $T = (t_{kl})$ with $t_{kl} = \frac{n^2}{4}\mathrm{tr}(\rho\lambda_k \otimes \lambda_l)$, $k, l = 1, 2, ..., n^2 - 1$. For a general $\mathcal{O} \otimes \mathcal{O}$ invariant state (16) we have,

$$\begin{aligned} x_k =& \frac{n}{2}\mathrm{tr}(\rho\lambda_k \otimes \mathbb{I}_n) \\ =& \frac{n}{2}[a\,(\mathrm{tr}\lambda_k)(\mathrm{tr}\mathbb{I}_n) + b\,\mathrm{tr}(\sum_{i,j} |i\rangle\langle j|\lambda_k \otimes |j\rangle\langle i|\mathbb{I}_n) + \\ & c\,\mathrm{tr}(\sum_{i,j} |i\rangle\langle j|\lambda_k \otimes |i\rangle\langle j|\mathbb{I}_n)] \end{aligned} \tag{27}$$

The first term is zero as the matrices λ_k's are traceless and the the coefficient of b in second term becomes,

$$\begin{aligned} \mathrm{tr}(\sum_{i,j} |i\rangle\langle j|\lambda_k \otimes |j\rangle\langle i|\mathbb{I}_n) =& \sum_{i,j} \mathrm{tr}(|i\rangle\langle j|\lambda_k)\mathrm{tr}(|j\rangle\langle i|\mathbb{I}_n) \\ =& \sum_{i,j} \langle j|\lambda_k|i\rangle.\delta_{ij} \\ =& \mathrm{tr}(\lambda_k) \\ =& 0 \end{aligned} \tag{28}$$

and similarly the coefficient of c in the third term becomes,

$$\begin{aligned} \mathrm{tr}(\sum_{i,j} |i\rangle\langle j|\lambda_k \otimes |i\rangle\langle j|\mathbb{I}_n) =& \sum_{i,j} \mathrm{tr}(|j\rangle\langle i|\lambda_k) tr(|i\rangle\langle j|\mathbb{I}_n) \\ =& \mathrm{tr}(\lambda_k) \\ =& 0 \end{aligned} \tag{29}$$

Hence, we have,

$$x_k = 0 \;\text{ for all}\, k = 1, 2, ..., , n^2 - 1 \tag{30}$$

The correlation matrix elements,

$$\begin{aligned} t_{kl} =& \frac{n^2}{4}\mathrm{tr}(\rho\lambda_k \otimes \lambda_l) \\ =& \frac{n^2}{4}\mathrm{tr}[a \sum_{i,j} |i\rangle\langle i|\lambda_k \otimes |j\rangle\langle j|\lambda_l + b \sum_{i,j} |i\rangle\langle j|\lambda_k \otimes |j\rangle\langle i|\lambda_l \\ & + c \sum_{i,j} |i\rangle\langle j|\lambda_k \otimes |i\rangle\langle j|\lambda_l] \\ =& \frac{n^2}{4}\mathrm{tr}[b \sum_{i,j} |i\rangle\langle j|\lambda_k \otimes |j\rangle\langle i|\lambda_l + c \sum_{i,j} |i\rangle\langle j|\lambda_k \otimes |i\rangle\langle j|\lambda_l] \end{aligned} \tag{31}$$

Whenever $k \neq l$,

$$\begin{aligned} t_{kl} =& \frac{n^2}{4}[b \sum_{i,j} \langle j|\lambda_k|i\rangle\langle i|\lambda_l|j\rangle + c \sum_{i,j} \langle j|\lambda_k|i\rangle\langle j|\lambda_l|i\rangle] \\ =& \frac{n^2}{4}[c \sum_{i,j} (\lambda_k)_{ji}(\lambda_l)_{ji}] = 0 \end{aligned} \tag{32}$$

By $(\lambda_k)_{ij}$ we denote the ij-th element of λ_k. The second equality follows from the fact that according to the construction of SU(n) generators (4), any two λ_k and $\lambda_l, k \neq l$ has no element in common at any position (or conjugate position) in their respective matrix form in computational basis.
Whenever $k = l$,

$$t_{kk} = \frac{n^2}{4}[2b + c \sum_{i,j} (\lambda_k)^2_{ji}] \tag{33}$$

There are n^2-1 generators of SU(n) and among them, $\sum_{i,j}(\lambda_k)^2_{ji} = 2$ for $k = 1, ..., \frac{n^2+n-2}{2}$ and $\sum_{i,j}(\lambda_k)^2_{ji} = -2$ for $k = \frac{n^2+n}{2}, ..., n^2-1$. Hence,

$$t_{kk} = \frac{n^2}{2}\begin{cases}(b+c) \text{ for } k = 1, 2, ..., \frac{n^2+n-2}{2}\\ (b-c) \text{ for } k = \frac{n^2+n}{2}, ..., n^2-1\end{cases} \tag{34}$$

Now, we can evaluate the bounds for the geometric discord and measurement-induced nonlocality from the relations (12), (13) as,

$$D(\rho) \geq \begin{cases}(n^2-n)(b^2+c^2), & \text{if } bc \geq 0\\ (n^2-n)(b^2+c^2)+4(n-1)bc, & \text{if } bc < 0\end{cases} \tag{35}$$

and

$$N(\rho) \leq \begin{cases}(n^2-n)(b^2+c^2), & \text{if } bc < 0\\ (n^2-n)(b^2+c^2)+4(n-1)bc, & \text{if } bc \geq 0\end{cases} \tag{36}$$

Since, $\mathbf{x} = \mathbf{0}$, the extra constraints (11) is automatically satisfied. Hence, discord and MIN becomes minimum and maximum value of the same optimization problem. So, $D(\rho) \leq N(\rho)$. It follows,

$$\begin{aligned} 0 &\leq (n^2-n)(b^2+c^2)+4(n-1)bc \leq D(\rho) \leq N(\rho) \leq (n^2-n)(b^2+c^2)\,, \text{ if } bc \leq 0\\ 0 &\leq (n^2-n)(b^2+c^2) \leq D(\rho) \leq N(\rho) \leq (n^2-n)(b^2+c^2)+4(n-1)bc\,, \text{ if } bc \geq 0\end{aligned} \tag{37}$$

Thus, we obtain bounds for both geometric discord and MIN for $\mathcal{O}\otimes\mathcal{O}$ invariant class of states. Clearly, the bounds saturate when at least one of b and c is zero. It is also interesting to note that whenever $b \neq 0$ or $c \neq 0$ the lower bounds are strictly positive. Hence, all $\mathcal{O}\otimes\mathcal{O}$ invariant class of states possess quantum correlation.

VII. DISCORD AND NEGATIVITY

Whenever, $b = 0$ or $c = 0$ the above $\mathcal{O}\otimes\mathcal{O}$ invariant class of states reduces to Isotropic and Werner classes respectively. The bounds become equal for these cases. For Werner class $D = \frac{(n-1)(1-n^2a)^2}{n}$ with $0 \leq a \leq \frac{1}{n^2-1}$ and for Isotropic class $D = \frac{(n-1)(1-n^2a)^2}{n}$, $\frac{1}{n(n+1)} \leq a \leq \frac{1}{n(n-1)}$. These results matches with the similar results in [13, 14]. Now, we choose a particular subclass of $\mathcal{O}\otimes\mathcal{O}$ invariant states with $a = \frac{1}{n^2}$. This class does not belongs to any of the above two classes. For this class, the other two parameters b and c must satisfy the relations $b+c = 0$ and other positivity constraints. For this class, $D \leq 2(n^2-n)b^2$. Partial transposition w.r.t. any party of a $\mathcal{O}\otimes\mathcal{O}$ invariant state results in interchange of b and c. So, for any state of $\mathcal{O}\otimes\mathcal{O}$ class, positivity conditions do not guarantee the positivity conditions of the partial transposed state. For $n = 3$, whenever $-\frac{1}{n^2} \leq b \leq -\frac{1}{n^2(n-1)}$ with $b+c = 0$, we get one negative eigenvalue of ρ^{T_A}. Logarithmic negativity of the transposed state can be evaluated using the definition $\mathcal{LN}(\rho) = \text{Log}_2(2\mathcal{N}+1)$ where $\mathcal{N}(\rho) = \frac{2}{n-1}\sum_{\lambda_i}|\lambda_i(\rho^{T_A})|$, λ_i denotes the negative eigenvalue of ρ^{T_A}. We have plotted our upper and lower bound of discord, log-negativity and squared negativity for the above case when $n = 3$. It reveals that discord can be less than log-negativity for this class, however $D > \mathcal{N}^2$ holds for this class.

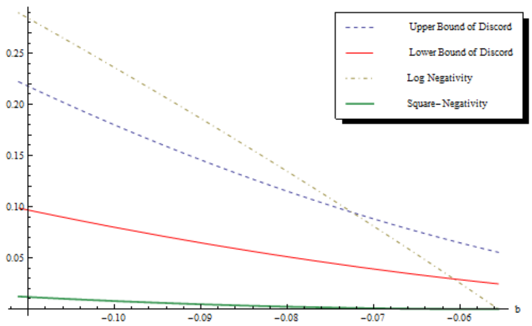

FIG. 4: (Color Online)*Discord and Negativity for a subclass of* $\mathcal{O}\otimes\mathcal{O}$ *invariant states with* $a = \frac{1}{n^2}$ *for* $n = 3$. *We choose the range of* b *as* $-\frac{1}{n^2} \leq b \leq -\frac{1}{n^2(n-1)}$ *and* $b+c = 0$. *Positivity constrains fix the range of* b *in* $[-\frac{1}{9}, \frac{1}{18}]$. *In this case the only negative eigenvalue of* ρ^{T_A} *is* $\frac{1}{n^2} + n\,b + c$. *We have used normalized version of discord. The normalization factor is* $\frac{n}{n-1}$.

VIII. CONCLUSION

We have calculated closed form of LQU and some bounds for geometric discord and measurement-induced

nonlocality for orthogonal invariant class of states. This result is then applied to check the explicit results of Werner and Isotropic class of states. Finally we have considered an important subclass of orthogonal invariant class. This subclass is different from Werner or Isotropic class. We checked discord for this class and compared it with some entanglement monotones such as Squared negativity, log-negativity. We obtain the value (upper bound) of discord for such states lower than the values of one of these entanglement monotones.

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